

INTRODUCING A PROBABILISTIC STRUCTURE ON SEQUENTIAL DYNAMICAL SYSTEMS, SIMULATION AND REDUCTION OF PROBABILISTIC SEQUENTIAL NETWORKS

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ABSTRACT. A probabilistic structure on sequential dynamical systems is introduced here, the new model will be called Probabilistic Sequential Network, PSN. The morphisms of Probabilistic Sequential Networks are defined using two algebraic conditions. It is proved here that two homomorphic Probabilistic Sequential Networks have the same equilibrium or steady state probabilities if the morphism is either an epimorphism or a monomorphism. Additionally, the proof of the set of PSN with its morphisms form the category **PSN**, having the category of sequential dynamical systems **SDS**, as a full subcategory is given. Several examples of morphisms, subsystems and simulations are given.

1. INTRODUCTION

Probabilistic Boolean Networks was introduced by I. Schmulevich, E. Dougherty, and W. Zhang in 2000, for studying the dynamic of a network using time discrete Markov chains, see [14, 15, 17, 16]. This model had several applications in the study of cancer, see [18]. It is important for development an algebraic mathematical theory of the model Probabilistic Boolean Network PBN, to describe special maps between two PBN, called homomorphism and projection, the first papers in this direction are, [4, 6], b. Instead of this model is being used in applications, the connection of the graph of genes and the State Space is an interesting problem to study. The introduction of probabilities in the definition of Sequential Dynamical System has this objective.

The theory of sequential dynamical systems (SDS) was born studying networks where the entities involved in the problem do not necessarily arrive at a place at the same time, and it is part of the theory of computer simulation, [2, 3]. The mathematical background for SDS was recently development by Laubenbacher and Pareigis, and it solves aspects of the theory and applications, see [8, 9, 10].

The introduction of a probabilistic structure on Sequential Dynamical Systems is an interesting problem that it is introduced in this paper. A SDS induces a finite dynamical system (k^n, f) , [5], but the mean difference between a SDS and FDS is that a SDS has a graph with information giving by the local functions, and an order in the sequential behavior of these local functions. It is known b₁, that a finite dynamical systems can be studied as a SDS, because we can construct a

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b We will use the acronym PBN, PSN, or SDS for plural as well as singular instances.

b₁Information giving by Laubenbacher.

bigger system that in this case is sequential. Making together the sequential order and the probabilistic structure in the dynamic of the system, the possibility to work in applications to genetics increase, because genes act in a sequential manner. In particular the notion of morphism in the category of SDS establishes connection between the digraph of genes and the State Space, that is the dynamic of the function. Working in the applications, Professor Dougherty's group wanted to consider two things in the definition of PBN: a sequential behavior on genes, and the exact definition of projective maps between two PBN that inherits the properties of the first digraph of genes. For this reason, a new model that considers both questions and tries to construct projections that work well is described here. I introduce in this paper the sequential behavior and the probability together in PSN and my final objective is to construct projective maps that let us reduce the number of functions in the finite dynamical systems inside the PBN. One of the mean problem in modeling dynamical systems is the computational aspect of the number of functions and the computation of steady states in the State Space. In particular, the reduction of number of functions is one of the most important problems, because by solving that we can determine which part of the network *State Space* may be simplified. The concept of morphism, simulation, epimorphism, and equivalent Probabilistic Sequential Networks are developed in this paper, with this particular objective.

This paper is organized as follows. In section 2, a notation slightly different to the one used in [9] is introduced for homomorphisms of SDS. This notation is helpful for giving the concept of morphism of PSN. In section 3, the probabilistic structure on SDS is introduced using for each vertex of the support graph, a set of local functions, more than one schedule, and finally having several update functions with probabilities assigned to them. So, it is obtained a new concept: probabilistic sequential network (PSN). In Theorem 4.3 is proved that monomorphisms, and epimorphisms of PSN have the same *equilibrium or steady state probabilities*. These strong results justify the introduction of the dynamical model PSN as an application to the study of sequential systems. In section 5, we prove that the PSN with its morphisms form the category **PSN**, having the category **SDS** as a full subcategory. Several examples of morphisms, subsystems and simulations are given in Section 6.

2. PRELIMINARIES

In this introductory section we give the definitions and results of Sequential Dynamical System introduced by Laubenbacher and Pareigis in [9]. Let Γ be a graph, and let $V_\Gamma = \{1, \dots, n\}$ be the set of vertices of Γ . Let $(k_i | i \in V_\Gamma)$ be a family of finite sets. The set k_a are called the set of local states at a , for all $a \in V_\Gamma$. Define $k^n := k_1 \times \dots \times k_n$ with $|k_i| < \infty$, the set of (global) states of Γ . A Sequential Dynamical System (SDS)

$$\mathcal{F} = (\Gamma, (k_i)_{i=1}^n, (f_i)_{i=1}^n, \alpha)$$

consists of

1. A finite graph $\Gamma = (V_\Gamma, E_\Gamma)$ with the set of vertices $V_\Gamma = \{1, \dots, n\}$, and the set of edges $E_\Gamma \subseteq V_\Gamma \times V_\Gamma$.
2. A family of finite sets $(k_i | i \in V_\Gamma)$.

3. A family of local functions $f_i : k^n \rightarrow k^n$, that is

$$f_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, \bar{f}, x_{i+1}, \dots, x_n)$$

where $\bar{f}(x_1, \dots, x_n)$ depends only of those variables which are connected to i in Γ .

4. A permutation $\alpha = (\alpha_1 \dots \alpha_n)$ in the set of vertices V_Γ , called an update schedule (i.e. a bijective map $\alpha : V_\Gamma \rightarrow V_\Gamma$).

The global update function of the SDS is $f = f_{\alpha_1} \circ \dots \circ f_{\alpha_n}$. The function f defines the dynamical behavior of the SDS and determines a finite directed graph with vertex set k^n and directed edges $(x, f(x))$, for all $x \in k^n$, called the State Space of \mathcal{F} , and denoted by \mathcal{S}_f .

The definition of homomorphism between two SDS uses the fact that the vertices $V_\Gamma = \{1, \dots, n\}$ of a SDS and the states k^n together with their evaluation map $k^n \times V_\Gamma \ni (x, a) \mapsto \langle x, a \rangle := x_a \in k_i$, form a contravariant setup, so that morphisms between such structures should be defined contravariantly, i.e. by a pair of certain maps $\phi : \Gamma \rightarrow \Delta$, and the induced function $h_\phi : k^m \rightarrow k^n$ with the graph Δ having m vertices. Here we use a notation slightly different that the one using in [9].

Let $F = (\Gamma, (f_i : k^n \rightarrow k^n), \alpha)$ and $G = (\Delta, (\hat{g}_i : k^m \rightarrow k^m), \beta)$ be two SDS. Let $\phi : \Delta \rightarrow \Gamma$ be a digraph morphism. Let $(\hat{\phi}_b : k_{\phi(b)} \rightarrow k_b, \forall b \in \Delta)$, be a family of maps in the category of **Set**. The map h_ϕ is an adjoint map, because is defined as follows: consider the pairing $k^n \times V_\Gamma \ni (x, a) \mapsto \langle x, a \rangle := x_a \in k_a$; and similarly $k^m \times V_\Delta \ni (y, b) \mapsto \langle y, b \rangle := y_b \in k_b$. The induced adjoint map holds $\langle h_\phi(x), b \rangle := \hat{\phi}_b(\langle x, \phi(b) \rangle) = \hat{\phi}_b(x_{\phi(b)})$. Then ϕ , and $(\hat{\phi}_b)$ induce the adjoint map $h_\phi : k^n \rightarrow k^m$ defined as follows:

$$(2.1) \quad h_\phi(x_1, \dots, x_n) = (\hat{\phi}_1(x_{\phi(1)}), \dots, \hat{\phi}_m(x_{\phi(m)})).$$

Then $h : F \rightarrow G$ is a homomorphism of SDS if for all sets of orders τ_β associated to β in the connected components of Δ , the map h_ϕ holds the following conditions:

$$(2.2) \quad (g_{\beta_l} \circ g_{\beta_{l+1}} \circ \dots \circ g_{\beta_s}) \circ h_\phi = h_\phi \circ f_{\alpha_i}$$

$$\begin{array}{ccc} k^n & \xrightarrow{f_{\alpha_i}} & k^n \\ h_\phi \downarrow & & h_\phi \downarrow \\ k^m & \xrightarrow{g_{\beta_l} \circ \dots \circ g_{\beta_s}} & k^m \end{array}$$

where $\{\beta_l, \beta_{l+1}, \dots, \beta_s\} = \phi^{-1}(\alpha_i)$. If $\phi^{-1}(\alpha_i) = \emptyset$, then $Id_{k^m} \circ h_\phi = h_\phi \circ f_{\alpha_i}$, and the commutative diagram is now the following:

$$(2.3) \quad \begin{array}{ccc} k^n & \xrightarrow{f_{\alpha_i}} & k^n \\ h_\phi \downarrow & & h_\phi \downarrow \\ k^m & \xrightarrow{Id_{k^m}} & k^m \end{array}$$

For examples and properties see[9]. It that paper, the authors proved that the above diagrams implies the following one

$$(2.4) \quad \begin{array}{ccc} k^n & \xrightarrow{f=f_{\alpha_1} \circ \dots \circ f_{\alpha_n}} & k^n \\ h_\phi \downarrow & & h_\phi \downarrow \\ k^m & \xrightarrow{g=g_{\beta_1} \circ \dots \circ g_{\beta_m}} & k^m \end{array}$$

Probabilistic Boolean Networks [14, 15, 17, 18] The model Probabilistic Boolean Network $\mathcal{A} = \mathcal{A}(\Gamma, F, C)$ is defined by the following:

- (1) a finite digraph $\Gamma = (V_\Gamma, E_\Gamma)$ with n vertices.
- (2) a family $F = \{F_1, F_2, \dots, F_n\}$ of ordered sets $F_i = \{f_{i1}, f_{i2}, \dots, f_{il(i)}\}$ of functions $f_{ij} : \{0, 1\}^n \rightarrow \{0, 1\}$, for $i = 1, \dots, n$, and $j = 1, \dots, l(i)$ called predictors,
- (3) and a family $C = \{c_{ij}\}_{i,j}$, of selection probabilities. The selection probability that the function f_{ij} is used for the vertex i is c_{ij} .

The dynamic of the model Probabilistic Boolean Network is given by the vector functions $\mathbf{f}_k = (f_{1k_1}, f_{2k_2}, \dots, f_{nk_n}) : \{0, 1\}^n \rightarrow \{0, 1\}^n$ for $1 \leq k_i \leq l(i)$, and $f_{ik_i} \in F_i$, acting as a transition function. Each variable $x_i \in \{0, 1\}$ represents the state of the vertex i . All functions are updated synchronously. At every time step, one of the functions is selected randomly from the set F_i according to a predefined probability distribution. The selection probability that the predictor f_{ij} is used to predict gene i is equal to

$$c_{ij} = P\{f_{ik_i} = f_{ij}\} = \sum_{k_i=j} p\{\mathbf{f} = f_k\}.$$

There are two digraph structures associated with a Probabilistic Boolean Network: the low-level graph Γ , and the high-level graph which consists of the states of the system and the transitions between states. The state space S of the network together with the set of network functions, in conjunction with transitions between the states and network functions, determine a Markov chain. The random perturbation makes the Markov chain ergodic, meaning that it has the possibility of reaching any state from another state and that it possesses a long-run (steady-state) distribution. As a Genetic Regulatory Network (GRN), evolves in time, it will eventually enter a fixed state, or a set of states, through which it will continue to cycle. In the first case the state is called a singleton or fixed point attractor, whereas, in the second case it is called a cyclic attractor. The attractors that the network may enter depend on the initial state. All initial states that eventually produce a given attractor constitute the basin of that attractor. The attractors represent the fixed points of the dynamical system that capture its long-term behavior. The number of transitions needed to return to a given state in an attractor is called the cycle length. Attractors may be used to characterize a cells phenotype (Kauffman, 1993) [7]. The attractors of a Probabilistic Genetic Regulatory Network (PGRN) are the attractors of its constituent GRN. However, because a PGRN constitutes an ergodic Markov chain, its steady-state distribution plays a key role. Depending on the structure of a PGRN, its attractors may contain most of the steady-state probability mass [1, 12, 19].

3. PROBABILISTIC SEQUENTIAL NETWORKS

The following definition give us the possibility to have several update functions acting in a sequential manner with assigned probabilities. All these, permit us to study the dynamic of these systems using Markov chains and other probability tools.

Definition 3.1. *A Probabilistic Sequential Network (PSN)*

$$\mathcal{D} = (\Gamma, \{F_a\}_{a=1}^{|\Gamma|=n}, (k_a)_{a=1}^n, (\alpha_j)_{j=1}^m, C = \{c_1, \dots, c_s\})$$

consists of:

- (1) a finite graph $\Gamma = (V_\Gamma, E_\Gamma)$ with n vertices;
- (2) a family of finite sets $(k_a | a \in V_\Gamma)$.
- (3) for each vertex a of Γ a set of local functions

$$F_a = \{f_{ai} : k^n \rightarrow k^n | 1 \leq i \leq \ell(i)\},$$

is assigned. (i. e. there exists a bijection map $\sim: V_\Gamma \rightarrow \{F_a | 1 \leq a \leq n\}$)
(for definition of local function see (??).2)).

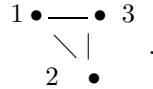
- (4) a family of m permutations $\alpha = (\alpha_1 \dots \alpha_n)$ in the set of vertices V_Γ .
- (5) and a set $C = \{c_1, \dots, c_s\}$, of assign probabilities to s update functions.

We select one function in each set F_a , that is one for each vertices a of Γ , and a permutation α , with the order in which the vertex a is selected, so there are \underline{n} possibly different update functions $f_i = f_{\alpha_1 i_1} \circ \dots \circ f_{\alpha_n i_n}$, where $\underline{n} \leq n! \times \ell(1) \times \dots \times \ell(n)$. The probabilities are assigned to the update functions, so there exists a set $S = \{f_1, \dots, f_s\}$ of selected update functions such that $c_i = p(f_i)$, $1 \leq i \leq s$.

Definition 3.2. The State Space of \mathcal{D} is a weighted digraph whose vertices are the elements of k^n and there is an arrow going from the vertex u to the vertex v if there exists an update function $f_i \in S$, such that $v = f_i(u)$. The probability $p(u, v)$ of the arrow going from u to v is the sum of the probabilities c_{f_i} of all functions f_i , such that $v = f_i(u)$, $u \xrightarrow{p(u,v)} f_i(u) = v$. We denote the State Space by $\mathcal{S}_\mathcal{D}$.

For each one update function in S we have one SDS inside the PSN, so the State Space \mathcal{S}_f is a subdigraph of $\mathcal{S}_\mathcal{D}$. When we take the whole set of update functions generated by the data, we will say that we have the *full* PSN. It is very clear that a SDS is a particular PSN, where we take one local function for each vertex, and one permutation. The dynamic of a PSN is described by Markov Chains of the transition matrix associated to the State Space.

Example 3.3. Let $\mathcal{D} = (\Gamma; F_1, F_2, F_3; \mathbf{Z}_2^3; \alpha_1, \alpha_2; (c_{f_i})_{i=1}^8)$, be the following PSN:



- (1) The graph Γ :

- (2) Let $\mathbf{x} = (x_1, x_2, x_3) \in \{0, 1\}^3$. In this paper, we always consider the operations over the finite field $\mathbb{Z}_2 = \{0, 1\}$, but we use additionally the following notation $\bar{x}_1 = x_1 + 1$. Then the sets of local functions from $\mathbb{Z}_2^3 \rightarrow \mathbb{Z}_2^3$ are the following

$$\begin{aligned} F_1 &= \{f_{11}(\mathbf{x}) = (1, x_2, x_3), f_{12}(\mathbf{x}) = (\bar{x}_1, x_2, x_3)\} \\ F_2 &= \{f_{21}(\mathbf{x}) = (x_1, x_1 x_2, x_3)\} \\ F_3 &= \{f_{31}(\mathbf{x}) = (x_1, x_2, x_1 x_2), f_{32}(\mathbf{x}) = (x_1, x_2, x_1 x_2 + x_3)\} \end{aligned}$$

- (3) The schedules or permutations are $\alpha_1 = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$; $\alpha_2 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$. We obtain the following table of functions, and we select all of them for \mathcal{D} because the probabilities given by C .

$$\begin{aligned} f_1 &= f_{31} \circ f_{21} \circ f_{11} & f_2 &= f_{11} \circ f_{21} \circ f_{31} \\ f_3 &= f_{32} \circ f_{21} \circ f_{11} & f_4 &= f_{11} \circ f_{21} \circ f_{32} \\ f_5 &= f_{31} \circ f_{21} \circ f_{12} & f_6 &= f_{12} \circ f_{21} \circ f_{31} \\ f_7 &= f_{32} \circ f_{21} \circ f_{12} & f_8 &= f_{12} \circ f_{21} \circ f_{32} \end{aligned}$$

The update functions are the following:

$$\begin{aligned} f_1(\mathbf{x}) &= (1, x_2, x_2) & f_2(\mathbf{x}) &= (1, x_1x_2, x_1x_2) \\ f_3(\mathbf{x}) &= (1, x_2, x_2 + x_3) & f_4(\mathbf{x}) &= (1, x_1x_2, x_1x_2 + x_3) \\ f_5(\mathbf{x}) &= (\bar{x}_1, \bar{x}_1x_2, (x_1 + 1)x_2) & f_6(\mathbf{x}) &= (\bar{x}_1, x_1x_2, x_1x_2) \\ f_7(\mathbf{x}) &= (\bar{x}_1, (x_1 + 1)x_2, (x_1 + 1)x_2 + x_3) & f_8(\mathbf{x}) &= (\bar{x}_1, x_1x_2, x_1x_2 + x_3) \end{aligned}$$

(4) The probabilities assigned are the following: $c_{f_1} = .18; c_{f_2} = .12; c_{f_3} = .18; c_{f_4} = .12; c_{f_5} = .12; c_{f_6} = .08; c_{f_7} = .12; c_{f_8} = .08$.

Example 3.4. We notice that there are several PSN that we can construct with the same initial data of functions and permutations, but with different set of probabilities, that is, subsystems of \mathcal{D} . For example if $S' = \{f_1, f_2, f_3, f_4\}$, $F'_1 = \{f_{11}\}$, and $D = \{d_{f_1} = .355, d_{f_2} = .211, d_{f_3} = .18, d_{f_4} = .254\}$, then

$$\mathcal{B} = (\Gamma; F'_1, F_2, F_3; \mathbf{Z}_2^3; \alpha_1, \alpha_2; D = \{.355, .211, .18, .254\}),$$

is a PSN too.

4. MORPHISMS OF PROBABILISTIC SEQUENTIAL NETWORKS

The definition of morphism of PSN is a natural extension of the concept of homomorphism of SDS. In this section we prove in Theorem 4.2 a strong property, that is the distribution of probabilities of two homomorphic PSN are enough close to prove Theorem 4.3.

Consider the following two PSN $\mathcal{D}_1 = (\Gamma, (F_a)_{a=1}^{|\Gamma|=n}, (k_a)_{a=1}^n, (\alpha^j)_j, C)$ and $\mathcal{D}_2 = (\Delta, (G_b)_{b=1}^{|\Delta|=m}, (k_b)_{b=1}^m, (\beta^j)_j, D)$. We denote by S_i the set of update functions of \mathcal{D}_i , $i = 1, 2$; and the following notation for $(u, v) \in k^n \times k^n$, and $(w, z) \in k^m \times k^m$,

$$c_f(u, v) = \begin{cases} p(f) & \text{if } f(u) = v \\ 0 & \text{otherwise} \end{cases}, \quad d_g(w, z) = \begin{cases} p(g) & \text{if } g(w) = z \\ 0 & \text{otherwise} \end{cases}$$

where $p(h)$ is the probability of the function h .

Definition 4.1. (Homomorphisms of PSN) A morphism $h : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ consist of:

- (1) A graph morphism $\phi : \Delta \rightarrow \Gamma$, and a family of maps in the category **Set**, $(\hat{\phi}_b : k_{\phi(b)} \rightarrow k_b \forall b \in \Delta)$, that induces the adjoint function h_ϕ , see (2.1).
- (2) The induced adjoint map $h_\phi : k^n \rightarrow k^m$ holds that for all update functions f in S_1 there exists an update function $g \in S_2$ such that h is a SDS-morphism from $(\Gamma, (f : k^n \rightarrow k^n), \alpha_j)$ to $(\Delta, (g : k^m \rightarrow k^m), \beta_j)$. That is, the diagrams 2.2, 2.3, and 2.4 commute for all f and its selected g .

$$(4.1) \quad \begin{array}{ccc} k^n & \xrightarrow{f=f_{\alpha_1} \circ \dots \circ f_{\alpha_n}} & k^n \\ h_\phi \downarrow & & \downarrow h_\phi \\ k^m & \xrightarrow{g=g_{\beta_1} \circ \dots \circ g_{\beta_m}} & k^m \end{array}$$

The second condition induces a map μ from S_1 to S_2 , that is $\mu(f) = g$ if the selected function for f is g . We say that a morphism h from \mathcal{D}_1 to \mathcal{D}_2 is a **PSN-isomorphism** if ϕ , h_ϕ , and μ are bijective functions, and $d(h_\phi(u), h_\phi(g(u))) = c(u, f(u))$ for all u , in k^n , and all $f \in S_1$, and all $g \in S_2$. We denote it by $\mathcal{D}_1 \cong \mathcal{D}_2$.

SPECIAL MORPHISMS. Let $\mathcal{D} = (\Gamma, (F_i)_{i=1}^n, (\alpha^j)_{j \in J}, C)$ be a PSN.

IDENTITY MORPHISM. The functions $\phi = id_\Gamma$, $h_\phi = id_{k^n}$, and $\mu = id_S$ define the *identity morphism* $\mathcal{I} : \mathcal{D} \rightarrow \mathcal{D}$, and it is a trivial example of a PSN-isomorphism.

MONOMORPHISM A morphism h of PSN is a *monomorphism* if ϕ is surjective, h_ϕ is injective, and for all f , and its associated g we have that $d_g \leq c_f$.

EPIMORPHISM A morphism is an *epimorphism* if ϕ is injective, h_ϕ is surjective, and for all f , and its associated g we have that $d_g \geq c_f$.

REMARK If the morphism h is either a monomorphism or an epimorphism, then the function μ is not necessary injective, neither surjective.

SOME THEOREMS

Theorem 4.2. *The morphism $h : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ induces the following probabilistic condition:*

For a fixed real number $0 \leq \epsilon < 1$, the map h_ϕ satisfies the following:

$$(4.2) \quad \max_{u,v} |c_f(u,v) - d_g(h_\phi(u), h_\phi(v))| \leq \epsilon$$

for all f in S_1 , and its selected g in S_2 , and all $(u,v) \in k^n \times k^n$.

Proof. Suppose ϕ , and h_ϕ satisfy the Definition 4.1; and

$$|c_f(u,v) - d_g(h_\phi(u), h_\phi(v))| \geq 1$$

for some $(u,v) \in k^n \times k^n$. Then we have one of the following cases

1. $c_f(u,v) = 1$ and $d_g(h_\phi(u), h_\phi(v)) = 0$. It is impossible by condition (2) in definition 4.1. In fact, if we have an arrow going from u to $v = f(u)$, then there exists an arrow going from $h_\phi(u)$ to $h_\phi(v) = g(h_\phi(u))$ by diagram 4.1, and the probability $d_g(h_\phi(u), h_\phi(v)) \neq 0$.
2. $c_f(u,v) = 0$, and $d_g(h_\phi(u), h_\phi(v)) = 1$. It is impossible because at least there exists one element $v_1 \in k^n$, such that $f(u) = v_1 \in k^n$ and $c_f(u, v_1) \neq 0$, then $d_g(h_\phi(u), h_\phi(v_1)) \neq 0$ too. Since the sum of probabilities of all arrow going up from $h_\phi(u)$ is equal 1, then $d_g(h_\phi(u), h_\phi(v)) < 1$, and our claim holds.

Therefore the condition holds, and always ϵ exists. \square

In the next theorem we will use the following notation:

- (1) $S_\phi = \mu(S_1)$.
- (2) $g^t = g \circ g \circ \dots \circ g$, t times.
- (3) $p_t(u,v) = \sum_{f^t} c_{f^t}(u,v)$, and $p_t(h_\phi(u), h_\phi(v)) = \sum_{g^t} d_{g^t}(h_\phi(u), h_\phi(v))$
- (4) T_i denotes the transition matrix of \mathcal{D}_i , for $i = 1, 2$, and $p_t(u,v) = (T_i^t)_{(u,v)}$.

Theorem 4.3. *If $h : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ is either a monomorphism or an epimorphism of probabilistic sequential networks, then:*

$$\lim_{t \rightarrow \infty} |(T_1^t)_{u,v} - (T_2^t)_{h_\phi(u), h_\phi(v)}| = 0,$$

for all $(u,v) \in k^n \times k^n$. That is, the equilibrium state of both systems are equals.

Proof. The condition giving by Theorem 4.2 asserts that, there exists a fixed real number $0 \leq \epsilon < 1$, such that the map h_ϕ satisfies the following:

$$\max_{u,v} |c_f(u,v) - d_g(h_\phi(u), h_\phi(v))| \leq \epsilon$$

for all f in S_1 , and its selected g in S_2 , and all $(u,v) \in k^n \times k^n$.

If there is a function f going from u to $v = f(u)$ in k^n , then there exists a function g going from $h_\phi(u)$ to $h_\phi(v)$, such that $g(h_\phi(u)) = h_\phi(f(u))$.

Because $c_{f^2}(u, f^2(u)) = c_f(u, f(u))c_f(f(u), f^2(u)) = c_f^2$, and

$$d_{g^2}(h_\phi(u), g^2(h_\phi(u))) = d_g(h_\phi(u), g(h_\phi(u)))d_g(g(h_\phi(u)), g^2(h_\phi(u))) = d_g^2.$$

We have

$$|c_{f^2}(u, f^2(u)) - d_{g^2}(h_\phi(u), g^2(h_\phi(u)))| = |c_f^2 - d_g^2|$$

If h is a monomorphism, then $c_f \geq d_g$, for all f and its associated g . Then

$$|c_{f^2}(u, f^2(u)) - d_{g^2}(h_\phi(u), g^2(h_\phi(u)))| = |c_f^2 - d_g^2| \leq c_f^2.$$

By induction we have that

$$|c_{f^t}(u, f^t(u)) - d_{g^t}(h_\phi(u), g^t(h_\phi(u)))| = |c_f^t - d_g^t| \leq c_f^t.$$

This result implies that

$$|p_t(u, v) - p_t(h_\phi(u), h_\phi(v))| \leq \sum_{f^t} c_f^t + \delta^t(u, v)$$

where $\delta^t(u, v) = \sum_{g \in \overline{G}(u, v)} d_g^t$, and $\overline{G}(u, v) = \{g \in G | g(h_\phi(u)) = h_\phi(v), \text{ and } g \neq \overline{\mu}(S_1)\}$.

Then, when t goes to infinity the sum $\sum_{f^t} c_f^t + \delta^t(u, v)$ goes to 0, and the theorem holds. If h is an epimorphism we obtain the same results, so the theorem holds again. \square

5. THE CATEGORY **PSN**

In this section, we prove that the PSN with the *morphisms* form a category, that we denote by **PSN**. For definitions, and results in Categories see [11].

Theorem 5.1. *Let $h_1 = (\phi_1, h_{\phi_1}) : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ and $h_2 = (\phi_2, h_{\phi_2}) : \mathcal{D}_2 \rightarrow \mathcal{D}_3$ be two morphisms of PSN. Then the composition $h = (\phi, h_\phi) = (\phi_2, h_{\phi_2}) \circ (\phi_1, h_{\phi_1}) = h_2 \circ h_1 : \mathcal{D}_1 \rightarrow \mathcal{D}_3$ is defined as follows: $h = (\phi, h_\phi) = (\phi_1 \circ \phi_2, h_{\phi_2} \circ h_{\phi_1})$ is a morphism of PSN. The function $\mu_h = \mu_{h_2} \circ \mu_{h_1}$.*

Proof. The composite function $\phi = \phi_1 \circ \phi_2$ of two graph morphisms is again a graph morphism. The composite function $h_\phi = h_{\phi_2} \circ h_{\phi_1}$ is again a digraph morphism which satisfies the conditions in Definition 4.1, by Proposition and Definition 2.7 in [9]. So, $h = (\phi, h_\phi)$ is again a morphism. of PSN. \square

Theorem 5.2. *The Probability Sequential Networks together with the homomorphisms of PSN form the category **PSN**.*

Proof. Easily follows from Theorem 5.1. \square

Theorem 5.3. *The SDS together with the morphisms defined in [9] form a full subcategory of the category **PSN**.*

Proof. It is trivial. \square

6. SIMULATION AND EXAMPLES

In this section we give several examples of morphisms, and simulations. In the second example we show how the Definition 4.1 is verified under the supposition that a function ϕ is defined. So, we have two examples in (6.2), one with ϕ the natural inclusion, and the second with ϕ a surjective map. The third, and the fourth examples are morphisms that represent simulation of \mathcal{G} by \mathcal{F} . We begin this section with the definitions of Simulation and sub-PSN.

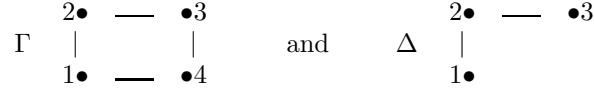
DEFINITION OF SIMULATION IN THE CATEGORY PSN. The probabilistic sequential network \mathcal{G} is simulated by \mathcal{F} if there exists a monomorphism $h : \mathcal{F} \rightarrow \mathcal{G}$ or an epimorphism $h' : \mathcal{G} \rightarrow \mathcal{F}$.

SUB PROBABILISTIC SEQUENTIAL NETWORK We say that a PSN \mathcal{G} is a sub Probabilistic Sequential Network of \mathcal{F} if there exists a monomorphism from \mathcal{G} to \mathcal{F} . If the map μ is not a bijection, then we say that it is a proper sub-PSN.

6.1. Examples.

(6.1.1) In the examples 3.3, and 3.4 we define two PSN \mathcal{D} and \mathcal{B} . The functions $\phi = Id_\Gamma$, $h_\phi = Id_{k^n}$, and μ the natural inclusion from S_1 to S_2 define the inclusion $\iota_\mu : \mathcal{B} \rightarrow \mathcal{D}$. It is clear that the inclusion is a monomorphism, so \mathcal{D} is simulated by \mathcal{B} .

(6.1.2) Consider the two graphs below



Suppose that the functions associated to the vertices are the families $\{f_1, f_2, f_3, f_4\}$, for Γ and $\{g_1, g_2, g_3\}$ for Δ . The permutations are $\alpha_1 = (4 \ 3 \ 2 \ 1)$, $\alpha_2 = (4 \ 1 \ 3 \ 2)$ and $\beta_1 = (3 \ 2 \ 1)$, $\beta_2 = (1 \ 3 \ 2)$, so, $S = \{f = f_4 \circ f_3 \circ f_2 \circ f_1; \underline{f} = f_1 \circ f_4 \circ f_3 \circ f_2\}$, and $X = \{g = g_3 \circ g_2 \circ g_1; \underline{g} = g_1 \circ g_3 \circ g_2\}$. Then, we have constructed two PSN, each one with two permutations and only one function associated to each vertex in the graph; denoted by:

$$\mathcal{D} = (\Gamma; f_1, f_2, f_3, f_4; \alpha_1, \alpha_2; C) \text{ and } \mathcal{B} = (\Delta; g_1, g_2, g_3; \beta_1, \beta_2; D).$$

Case (a) We assume that there exists a morphism $h : \mathcal{D} \rightarrow \mathcal{B}$, with the graph morphism $\phi : \Delta \rightarrow \Gamma$ giving by $\phi(1) = 1$, $\phi(2) = 2$, $\phi(3) = 3$. Suppose the functions

$$(\hat{\phi}_b : k_{\phi(b)} \rightarrow k_b, \forall b \in \Delta),$$

are giving, and the adjoint function

$$h_\phi : k^4 \rightarrow k^3, \quad h_\phi(x_1, x_2, x_3, x_4) = (\hat{\phi}_1(x_1), \hat{\phi}_2(x_2), \hat{\phi}_3(x_3))$$

is defined too. If h is a morphism, which satisfies the definition (4.1), then the following diagrams commute:

$$\begin{array}{ccccccccc} k^4 & \xrightarrow{f_4} & k^4 & \xrightarrow{f_3} & k^4 & \xrightarrow{f_2} & k^4 & \xrightarrow{f_1} & k^4 \\ h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow \\ k^3 & \xrightarrow{Id} & k^3 & \xrightarrow{g_3} & k^3 & \xrightarrow{g_2} & k^3 & \xrightarrow{g_1} & k^3 \end{array},$$

$$\begin{array}{ccccccccc}
k^4 & \xrightarrow{f_1} & k^4 & \xrightarrow{f_4} & k^4 & \xrightarrow{f_3} & k^4 & \xrightarrow{f_2} & k^4 \\
h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow \\
k^3 & \xrightarrow{g_1} & k^3 & \xrightarrow{Id} & k^3 & \xrightarrow{g_3} & k^3 & \xrightarrow{g_2} & k^3 \\
\\
k^4 & \xrightarrow{f} & k^4 & & k^4 & \xrightarrow{\underline{f}} & k^4 & & \\
h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow & & \\
k^3 & \xrightarrow{g} & k^3 & & k^3 & \xrightarrow{\underline{g}} & k^3 & &
\end{array}$$

Case (b) Consider now the map $\phi : \Gamma \rightarrow \Delta$, defined by $\phi(1) = 1$, $\phi(2) = 2$, $\phi(3) = 3$, and $\phi(4) = 3$. If there exists a morphism $h : \mathcal{B} \rightarrow \mathcal{D}$ that satisfies Definition 4.1, then

$$\begin{array}{ccccccc}
k^3 & \xrightarrow{g_3} & k^3 & \xrightarrow{g_2} & k^3 & \xrightarrow{g_1} & k^3 \\
h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow \\
k^4 & \xrightarrow{f_4 \circ f_3} & k^4 & \xrightarrow{f_2} & k^4 & \xrightarrow{f_1} & k^3 \\
k^3 & \xrightarrow{g_1} & k^3 & \xrightarrow{g_3} & k^3 & \xrightarrow{g_2} & k^3 \\
h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow \\
k^4 & \xrightarrow{f_1} & k^4 & \xrightarrow{f_4 \circ f_3} & k^4 & \xrightarrow{f_2} & k^3
\end{array}$$

(6.1.3) We now construct a monomorphism $h : \mathcal{F} \rightarrow \mathcal{G}$, with the properties that ϕ is surjective and the function h_ϕ is injective. The PSN $\mathcal{F} = (\Gamma, (F_i)_3, \beta, C)$ has the support graph Γ with three vertices, and the PSN $\mathcal{G} = (\Delta, (G_i)_4, \alpha, D)$ has the support graph Δ with four vertices

$$\begin{array}{ccc}
\Gamma & & \Delta \\
\begin{array}{ccc} & \bullet 3 & \\ 1 \bullet & \text{---} & \bullet 2 \end{array} & & \begin{array}{ccc} 2 \bullet & \text{---} & \bullet 4 \\ & \searrow & \downarrow \\ 1 \bullet & \text{---} & \bullet 3 \end{array}
\end{array}$$

The morphism $h : \mathcal{F} \rightarrow \mathcal{G}$, has the contravariant graph morphism $\phi : \Delta \rightarrow \Gamma$, defined by the arrows of graphs, as follows $\phi(1) = 1$, $\phi(2) = \phi(3) = 2$, and $\phi(4) = 3$, so it is a surjective map. The family of functions $\hat{\phi}_i : k_{\phi(i)} \rightarrow k_{(i)}$, $\hat{\phi}_1(x_1) = x_1$; $\hat{\phi}_2(x_2) = x_2$; $\hat{\phi}_3(x_2) = x_2$; $\hat{\phi}_4(x_4) = x_4$, are injective functions. The sets $k_a = \mathbb{Z}_2$, for all vertices a in Δ , and Γ . The adjoint function is $h_\phi : \mathbf{Z}_2^3 \rightarrow \mathbf{Z}_2^4$, defined by

$$h_\phi(x_1, x_2, x_3) = (\hat{\phi}_1(x_1), \hat{\phi}_2(x_2), \hat{\phi}_3(x_2), \hat{\phi}_4(x_4)) = (x_1, x_2, x_2, x_3).$$

Then, the first condition in the definition 4.1 holds.

The PSN $\mathcal{F} = (\Gamma; (F_i)_3; \beta; C)$, is defined with the following data. The set of functions $F_1 = \{f_{11}, f_{12}\}$, $F_2 = \{f_{21}\}$, and $F_3 = \{f_{31}\}$, where

$$f_{11} = Id, \quad f_{12}(x_1, x_2, x_3) = (1, x_2, x_3), \quad f_{21} = Id,$$

$$f_{31}(x_1, x_2, x_3) = (x_1, x_2, x_2 \overline{x_3}).$$

A permutation $\beta = (1 \ 2 \ 3)$; and the probabilities $C = \{c_{f_1} = .5168, c_{f_2} = .4832\}$. So, we are taking all the update functions $S = \{f_1, f_2\}$;

$$f_1 = f_{11} \circ f_{21} \circ f_{31}, \quad f_1(x_1, x_2, x_3) = (x_1, x_2, x_2 \overline{x_3});$$

$$\text{and } f_2 = f_{12} \circ f_{21} \circ f_{31}, \quad f_2(x_1, x_2, x_3) = (1, x_2, x_2 \overline{x_3}).$$

On the other hand, the PSN $\mathcal{G} = (\Delta; (G_i)_4; \alpha; D)$ has the following data. The families of functions: $G_1 = \{g_{11}, g_{12}\}$; $G_2 = \{g_{21}, g_{22}\}$, $G_3 = \{g_{31}, g_{32}\}$; and $G_4 = \{g_4\}$, where

$$\begin{aligned} g_{11}(x_1, x_2, x_3, x_4) &= (1, x_2, x_3, x_4) \\ g_{21}(x_1, x_2, x_3, x_4) &= (x_1, 1, x_3, x_4) & g_{12} = Id = g_{22} \\ g_{31}(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_1x_2, x_4) & g_{32}(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_2, x_4) \\ g_{41}(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_3, x_2\overline{x_4}) \end{aligned}$$

One permutation or schedule $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$. The assigned probabilities $d_{g_5} = .00252$, $d_{g_6} = .08321$, $d_{g_7} = .51428$, $d_{g_8} = .39999$ whose determine the set of update functions $X = \{g_5, g_6, g_7, g_8\}$: therefore the all update functions are the following

$$\begin{aligned} g_1 &= g_{11} \circ g_{21} \circ g_{31} \circ g_{41}, & g_2 &= g_{12} \circ g_{21} \circ g_{32} \circ g_{41} & g_3 &= g_{12} \circ g_{21} \circ g_{31} \circ g_{41}, \\ g_4 &= g_{11} \circ g_{21} \circ g_{32} \circ g_{41} & g_5 &= g_{12} \circ g_{22} \circ g_{31} \circ g_{41}, & g_6 &= g_{11} \circ g_{22} \circ g_{31} \circ g_{41} \\ g_7 &= g_{12} \circ g_{22} \circ g_{32} \circ g_{41}, & g_8 &= g_{11} \circ g_{22} \circ g_{32} \circ g_{41} \end{aligned}$$

The selected functions are

$$\begin{aligned} g_5(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_1x_2, x_2\overline{x_4}), & g_6(x_1, x_2, x_3, x_4) &= (1, x_2, x_1x_2, x_2\overline{x_4}) \\ g_7(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_2, x_2\overline{x_4}), & g_8(x_1, x_2, x_3, x_4) &= (1, x_2, x_2, x_2\overline{x_4}) \end{aligned}$$

We claim that $h : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism, in fact the following diagrams commute.

$$\begin{array}{ccc} \mathbb{Z}_2^3 & \xrightarrow{f_1} & \mathbb{Z}_2^3 \\ h_\phi \downarrow & & h_\phi \downarrow \\ \mathbb{Z}_2^4 & \xrightarrow{g_7} & \mathbb{Z}_2^4 \end{array}, \text{ and } \begin{array}{ccc} \mathbb{Z}_2^3 & \xrightarrow{f_2} & \mathbb{Z}_2^3 \\ h_\phi \downarrow & & h_\phi \downarrow \\ \mathbb{Z}_2^4 & \xrightarrow{g_8} & \mathbb{Z}_2^4 \end{array}.$$

In fact, $(h_\phi \circ f_1)(x_1, x_2, x_3) = (x_1, x_2, x_2, x_2\overline{x_3}) = (g_7 \circ h_\phi)(x_1, x_2, x_3)$, on the other hand $(h_\phi \circ f_2)(x_1, x_2, x_3) = (1, x_2, x_2, x_2\overline{x_3}) = (g_8 \circ h_\phi)(x_1, x_2, x_3)$ so, the property holds. We verify the second property in the definition of morphism for the compositions f_1 and g_7 , and also with the compositions f_2 and g_8 . That is, we check the sequence of local functions too.

$$\begin{array}{ccccccc} \mathbb{Z}_2^3 & \xrightarrow{f_{31}} & \mathbb{Z}_2^3 & \xrightarrow{f_{21}} & \mathbb{Z}_2^3 & \xrightarrow{f_{11}} & \mathbb{Z}_2^3 \\ h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow \\ \mathbb{Z}_2^4 & \xrightarrow{g_{41}} & \mathbb{Z}_2^4 & \xrightarrow{g_{22} \circ g_{32}} & \mathbb{Z}_2^4 & \xrightarrow{g_{12}} & \mathbb{Z}_2^3 \\ \mathbb{Z}_2^3 & \xrightarrow{f_{31}} & \mathbb{Z}_2^3 & \xrightarrow{f_{21}} & \mathbb{Z}_2^3 & \xrightarrow{f_{12}} & \mathbb{Z}_2^3 \\ h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow & & h_\phi \downarrow \\ \mathbb{Z}_2^4 & \xrightarrow{g_{41}} & \mathbb{Z}_2^4 & \xrightarrow{g_{22} \circ g_{32}} & \mathbb{Z}_2^4 & \xrightarrow{g_{11}} & \mathbb{Z}_2^3 \end{array}$$

$$\begin{aligned} (h_\phi \circ f_{31})(x_1, x_2, x_3) &= (x_1, x_2, x_2, x_2\overline{x_3}) = (g_{41} \circ h_\phi)(x_1, x_2, x_3), \\ (h_\phi \circ f_{21})(x_1, x_2, x_3) &= (x_1, x_2, x_2, x_3) = ((g_{22} \circ g_{32}) \circ h_\phi)(x_1, x_2, x_3), \\ (h_\phi \circ f_{11})(x_1, x_2, x_3) &= (x_1, x_2, x_2, x_3) = (g_{12} \circ h_\phi)(x_1, x_2, x_3), \\ (h_\phi \circ f_{12})(x_1, x_2, x_3) &= (1, x_2, x_2, x_3) = (g_{11} \circ h_\phi)(x_1, x_2, x_3) \end{aligned}$$

The probabilities satisfy the following conditions: $p(f_1) \geq p(g_7)$, and $p(f_2) \geq p(g_8)$. Then our claim holds, and h_ϕ is a monomorphism.

(6.1.4) We can construct an epimorphism $h' : \mathcal{G} \rightarrow \mathcal{F}$, that is, the function ϕ is injective and the function h'_ϕ is surjective. We use $\phi' : \Gamma \rightarrow \Delta$, defined as follow

$\phi'(i) = i + 1$, for all $i \in V_\Gamma$. Therefore $\hat{\phi}'_i : k_{\phi'(i)} \rightarrow k_{(i)}$, $\hat{\phi}'_i : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$, for all $i \in V_\Gamma$, and should be satisfies $\langle h_\phi(x), i \rangle := \hat{\phi}_b(\langle x, \phi(i) \rangle) = \hat{\phi}_b(x_{\phi(i)})$. So, the adjoint function is $h'_\phi(x_1, x_2, x_3, x_4) = (\hat{\phi}'_1(x_1), \hat{\phi}'_2(x_3), \hat{\phi}'_3(x_3)) = (x_1, x_2, x_4)$ and satisfies the following commutative diagrams

$$\begin{array}{ccccccc} \mathbb{Z}_2^4 & \xrightarrow{g_5} & \mathbb{Z}_2^4 & \mathbb{Z}_2^4 & \xrightarrow{g_7} & \mathbb{Z}_2^4 & \mathbb{Z}_2^4 & \xrightarrow{g_6} & \mathbb{Z}_2^4 & \mathbb{Z}_2^4 & \xrightarrow{g_8} & \mathbb{Z}_2^4 \\ h'_\phi \downarrow & & h'_\phi \downarrow, & h'_\phi \downarrow & & h'_\phi \downarrow, & h'_\phi \downarrow & \text{and} & h'_\phi \downarrow & & h'_\phi \downarrow \\ \mathbb{Z}_2^3 & \xrightarrow{f_1} & \mathbb{Z}_2^3 & \mathbb{Z}_2^3 & \xrightarrow{f_1} & \mathbb{Z}_2^3 & \mathbb{Z}_2^3 & \xrightarrow{f_2} & \mathbb{Z}_2^3 & \mathbb{Z}_2^3 & \xrightarrow{f_2} & \mathbb{Z}_2^3 \end{array}$$

These implies that $\mu(g_5) = \mu(g_7) = f_1$, and $\mu(g_6) = \mu(g_8) = f_2$. In fact,

$$\begin{aligned} (h'_\phi \circ g_5)(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_2\bar{x}_4) = (f_1 \circ h'_\phi)(x_1, x_2, x_3, x_4), \\ (h'_\phi \circ g_6)(x_1, x_2, x_3, x_4) &= (1, x_2, x_2\bar{x}_4) = (f_2 \circ h'_\phi)(x_1, x_2, x_3, x_4), \\ (h'_\phi \circ g_7)(x_1, x_2, x_3, x_4) &= (x_1, x_2, x_2\bar{x}_4) = (f_1 \circ h'_\phi)(x_1, x_2, x_3, x_4), \\ (h'_\phi \circ g_8)(x_1, x_2, x_3, x_4) &= (1, x_2, x_2\bar{x}_4) = (f_2 \circ h'_\phi)(x_1, x_2, x_3, x_4). \end{aligned}$$

Checking the compositions of local functions $g_5 = g_{12} \circ g_{22} \circ g_{31} \circ g_{41}$, and $f_1 = f_{11} \circ f_{21} \circ f_{31}$, we have that the following diagrams commute

$$\begin{array}{ccccccc} \mathbb{Z}_2^4 & \xrightarrow{g_{12}} & \mathbb{Z}_2^4 & \xrightarrow{g_{22}} & \mathbb{Z}_2^4 & \xrightarrow{g_{31}} & \mathbb{Z}_2^4 & \xrightarrow{g_{41}} & \mathbb{Z}_2^4 \\ h'_\phi \downarrow & & h'_\phi \downarrow & & h'_\phi \downarrow & & h'_\phi \downarrow & & h'_\phi \downarrow \\ \mathbb{Z}_2^3 & \xrightarrow{f_{11}} & \mathbb{Z}_2^3 & \xrightarrow{Id} & \mathbb{Z}_2^3 & \xrightarrow{f_{21}} & \mathbb{Z}_2^3 & \xrightarrow{f_{31}} & \mathbb{Z}^2 \end{array}$$

By the data we only need to check the following compositions

$h'_\phi(g_{31}(x_1, x_2, x_3, x_4)) = (x_1, x_2, x_4) = f_{21}(h'_\phi(x_1, x_2, x_3, x_4))$,
 $h'_\phi(g_{41}(x_1, x_2, x_3, x_4)) = (x_1, x_2, x_2\bar{x}_4) = f_{31}(h'_\phi(x_1, x_2, x_3, x_4))$. Similarly, we can prove that the other functions hold the property. The probabilities satisfy the following conditions: $p(g_5) \leq p(f_1), p(g_7) \leq p(f_1), p(g_6) \leq p(f_2)$, and $p(g_8) \leq p(f_2)$. Therefore h'_ϕ is an epimorphism.

7. EQUIVALENT PROBABILISTIC SEQUENTIAL NETWORKS

Definition 7.1. (Equivalent PSN) *If the morphism $h : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ satisfies that ϕ , h_ϕ and μ are bijective functions, but the probabilities are not necessary equals, we say that \mathcal{D}_1 , and \mathcal{D}_2 are equivalent PSN. We write $\mathcal{D}_1 \simeq \mathcal{D}_2$.*

So, \mathcal{D}_1 , and \mathcal{D}_2 are equivalents if there exist (ϕ, h_ϕ, μ) , and $(\phi^{-1}, h_\phi^{-1}, \mu^{-1})$, such that for all update functions $f \in \mathcal{D}_1$ and its selected function $g \in \mathcal{D}_2$, the condition $f = h_\phi^{-1} \circ g \circ h_\phi$ holds. It is clear that this relation is an equivalence relation in the set of PSN.

Proposition 7.2. *If $\mathcal{D}_1 \simeq \mathcal{D}_2$, then the transition matrices T_1 and T_2 satisfy: $(T_1^m)_{(u,v)} \neq 0$, if and only if $(T_2^m)_{(h_\phi(u), h_\phi(v))} \neq 0$, for all $m \in \mathbb{N}$, $(u, v) \in k^n \times k^n$.*

Proof. It is obvious. \square

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REFERENCES

- [1] Maria A. Avino, *Homomorphism of Probabilistic Gene Regulatory Networks*, Proceedings of Workshop on Genomic Signal Processing and Statistics (GENSIPS) 2006, TX, 4 pages.
- [2] C. Barret, and C. Reidys, *Elements of a theory of simulation I: Sequential CA Over Random Graphs*, Appl. Math. Comput. **98**pp 241-259, (1999).
- [3] C. Barret, H. Mortveit, and C. Reidys, *Elements of a theory of simulation II: sequential dynamical systems*, Appl. Math. Comput. **107** (2000).
- [4] E. R. Dougherty and I. Shmulevich, *Mappings between probabilistic Boolean networks*, Signal Processing, vol. 83, no. 4, pp. 799809, 2003.
- [5] Rene Hernandez Toledo, *Linear Finite Dynamical Systems*, Communications in Algebra, 33: 29772989, 2005 Copyright Taylor and Francis, Inc. ISSN: 0092-7872 print/1532-4125 online DOI: 10.1081/AGB-200066211.
- [6] I. Ivanov, and Edward R. Dougherty, *Reduction Mappings between Probabilistic Boolean Networks*, EURASIP Journal on Applied Signal Processing 2004:1, 125131
- [7] Kauffman, S.A. *The Origins of Order: Self-organization and Selection in Evolution*. Oxford University Press, NY, (1993)
- [8] R. Laubenbacher and B. Paregis, *Equivalence relations on finite dynamical systems*. Adv, in Appl. Math **26** (2001), 237-251.
- [9] R. Laubenbacher and B. Paregis, *Decomposition and simulation of sequential dynamical systems* Advances in Applied Mathematics, 30 (655-678) 2003.
- [10] R. Laubenbacher, B. Paregis (2006) *Update schedules of sequential dynamical systems*. Discrete Applied Mathematics 154: 980-994.
- [11] S. MacLane *Categories for the Working Mathematicians* Springer-Verlag New York Inc., 1971.
- [12] S. Marshall, L. Yu, Y. Xiao, and Edward R. Dougherty, *Inference of a Probabilistic Boolean Network from a Single Observed Temporal Sequence*, Hindawi Publishing Corporation EURASIP Journal on Bioinformatics and Systems Biology Volume 2007, Article ID 32454, 15 pages.
- [13] W. J. Stewart, *Introduction to the Numerical Solution of Markov Chains* Princeton University Press, 1994.
- [14] I. Shmulevich, E. R. Dougherty, S. Kim, and W. Zhang, *Probabilistic Boolean networks: a rule-based uncertainty model for gene regulatory networks*, Bioinformatics 18(2):261-274, (2002).
- [15] I. Shmulevich, E. R. Dougherty, and W. Zhang, *Gene perturbation and intervention in probabilistic Boolean networks*, Bioinformatics 18(10):1319-1331, (2002).
- [16] I. Shmulevich, E. R. Dougherty, and W. Zhang, *Control of stationary behavior in probabilistic Boolean networks by means of structural intervention*, J. Biol. Systems 10 (4) (2002) 431-445.
- [17] I. Shmulevich, E. R. Dougherty, and W. Zhang, *From Boolean to probabilistic Boolean networks as models of genetic regulatory networks*, Proceedings of the IEEE, vol. 90, no. 11, pp. 17781792, 2002.
- [18] I. Shmulevich¹, I. Gluhovsky², R. F. Hashimoto E. R. Dougherty, and W. Zhang, *Steady-state analysis of genetic regulatory networks modelled by probabilistic Boolean networks*, Comparative and Functional Genomics, Comp Funct Genom 2003; 4: 601608. Published online in Wiley InterScience.
- [19] X. Zhou¹, X. Wang, R. Pal¹, I. Ivanov, M. Bittner, and Edward R. Dougherty, *A Bayesian connectivity-based approach to constructing probabilistic gene regulatory networks*, BIOINFORMATICS, Vol. 20 no. 17 2004, pages 29182927.

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